GAUSSIAN PROCESSES
EXERCISE SHEET 0: FRAMEWORK OF PROBABILITY AND SOME BASIC CALCULATIONS

Exercise 1.

Assume that X has characteristic function $\varphi_X(t) = \exp(-C|t|^\alpha)$ for some $C > 0$. Then if $X_1, \ldots, X_n \sim X$, we have $\varphi_{b_n^{-1}(X_1+\cdots+X_n)}(t) = \exp(-nCb_n^{-\alpha}|t|^\alpha)$, which coincides with $\varphi_X(t)$ when one chooses $b_n = n^{1/\alpha}$. □

Exercise 2.

Assume that X has a (strictly) stable law. Then if $Y_1, \ldots, Y_n \sim X$ are independent, we have $Y_1 + \cdots + Y_n \sim cX$ for some $c > 0$. Defining $X_j := \frac{Y_j}{c}$ proves the first claim.

An example ofan infinitely divisible but not stable probability law is the Poisson distribution. If X is Poisson with parameter λ, i.e. $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$, then one sees that $X \sim X_1 + \cdots + X_n$ where X_j are i.i.d. Poisson random variables with parameter λ/n. However, if, on the other hand $X_1, X_2 \sim X$ are independent then $X_1 + X_2$ has a Poisson probability law with parameter 2λ, which is not of the form cX. This is because for cX to be integer valued and be able to take on every non-negative integer one has to have $c = 1$. Finally, Poisson random variables have finite variance λ. □

Exercise 3.

The characteristic function:

$$\varphi_X(t) = \int_{\mathbb{R}} \frac{1}{\pi} \frac{1}{1+x^2} e^{itx} dx = e^{-|t|}. \tag{0.1}$$

Thus by Exercise 1, $b_n = n$. □

Exercise 4.

Let $X_1, \ldots, X_n, X_1', \ldots, X_m'$ be i.i.d. with the same law as X. Then

$$(n+m)^{1/\alpha} X \sim X_1 + \cdots + X_n + X_1' + \cdots + X_m' \sim n^{1/\alpha} X + m^{1/\alpha} X',$$

from which we see that

$$a(1 + \frac{m}{n})^{1/\alpha} X \sim aX + a\Big(\frac{m}{n}\Big)^{1/\alpha} X'.$$

Letting $m/n \to b^\alpha/a^\alpha$ as $m, n \to \infty$ we deduce (e.g. by using Levy's continuity theorem) that

$$aX + bX' \sim (a^\alpha + b^\alpha)^{1/\alpha} X.$$

□

Exercise 5.

As in the solution of Exercise 5, we take $Z = X - X'$ where X' has the same law as X and is independent of X. If X is discrete, Z would also be discrete. On the other hand, we already know that $\varphi_Z(t) = e^{-c|t|^\alpha}$ for some $c > 0$ and $0 < \alpha \leq 2$. Note that the function $\varphi_Z(t) = e^{-c|t|^\alpha}$ is continuous and absolutely integrable, by inverse Fourier transform, we have that Z is continuous and the density function of Z is

$$p_Z(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \varphi_Z(t)e^{-itx}dt, \tag{0.2}$$

which means that we have a contradiction. □

Exercise 6.

The idea is to notice that $\exp(-C|t|^\alpha)$ has vanishing second derivative at 0. This at least formally implies that $\mathbb{E}X^2 = 0$, which is not possible.

To make this precise, let us look at $\frac{2-e^{itX}-e^{-itX}}{t^2}$. For any X and t this is a non-negative real number and moreover as $t \to 0$ it tends to X^2. Hence by Fatou's lemma

$$\mathbb{E}X^2 = \mathbb{E}\lim_{t\to 0}\frac{2-e^{itX}-e^{-itX}}{t^2} \leq \liminf_{t\to 0}\mathbb{E}\frac{2-e^{itX}-e^{-itX}}{t^2} = \liminf_{t\to 0}\frac{2-2e^{-C|t|^\alpha}}{t^2} = 0.$$

□

Exercise 7.

If X has a strictly stable law, then its characteristic function φ_X satisfies the following functional equation:

$$\varphi_X(t)^k = \varphi_X(b_k t)$$

Consider now a variable $Z = X - X'$ where X' has the same law as X and is independent of X. Then $\varphi_Z(t) = |\varphi_X(t)|^2$ is non-negative and satisfies also

$$\varphi_Z(t)^k = \varphi_Z(b_k t).$$

From here one easily deduces by induction that

$$\varphi_Z(b_{k_1}^{n_1}\dots b_{k_m}^{n_m}t) = \varphi_Z(t)^{k_1^{n_1}\dots k_m^{n_m}}$$

for all $t \in \mathbb{R}$, $k_1, \dots, k_m \geq 1$ and $n_1, \dots, n_m \in \mathbb{Z}$. Assuming X is not a.s. constant (this case is easy to check separately) we can find $t \in \mathbb{R}$ such that $0 < |\varphi_X(t)| < 1$. Fix such t and $k \geq 3$ and consider the equation

$$\varphi_Z(b_2^n b_k^m t) = \varphi_Z(t)^{2^n k^m}$$

for $m = \left[-\frac{n\log(b_2)}{\log(b_k)}\right]$, where $[x]$ denotes the nearest integer to $x \in \mathbb{R}$. One can find a sequence of n such that $n \to \infty$ and $\left[-\frac{n\log(b_2)}{\log(b_k)}\right] + \frac{n\log(b_2)}{\log(b_k)} \to 0$. Taking the limit in the equation above along such sequence we get

$$\varphi_Z(t) = \lim_n \varphi_Z(t)^{2^n k^m}.$$

Since $|\varphi_Z(t)| < 1$, the exponent $2^n k^m$ must stay bounded, which implies that

$$\frac{\log(b_2)}{\log(b_k)} = \frac{\log(2)}{\log(k)}.$$

Note that $\varphi_Z(0) = 1$ and X is not a constant, thus we can choose $\alpha > 0$ so that $b_2 = 2^{1/\alpha}$, we see that $b_k = k^{\log(b_2)/\log(2)} = k^{1/\alpha}$. Inserting this into the equation we had in the beginning, we get

$$\varphi_Z(k_1^{n_1/\alpha}\dots k_m^{n_m/\alpha}) = \varphi_Z(1)^{k_1^{n_1}\dots k_m^{n_m}}$$

and approximating arbitrary $t > 0$ with rationals we get

$$\varphi_Z(t^{1/\alpha}) = \varphi_Z(1)^t,$$

or equivalently

$$\varphi_Z(t) = \varphi_Z(1)^{t^\alpha}.$$

Furthermore, note that $\varphi_Z(-1) = \overline{\varphi_Z(1)}$ and $|\varphi_Z(t)| \le 1$, it follows that $\varphi_Z(t) = e^{-c|t|^\alpha}$ for some $c > 0$. By Ex. 6, we must have $\alpha \le 2$.

$\square$